Knots, Links and Spatial Graphs

Spring 2020

Yangxinyu Xie

Last updated April 29, 2020

1. Introduction

In 1983, John Conway and Cameron Gordon published a well-known paper *Knots and links in spatial graph* [CMG83] that stimulated a wide range of interests in the study of spatial graph theory. In this note, we review this classical paper and some of the earliest results in intrinsic properties of graphs. We first review some of the definitions.

Definition 1.1 (Graphs)**.** A **graph** is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. The elements of V are the **vertices** of the graph G, and the elements of E are called **edges** of G. Two vertices $u, v \in V$ are called **adjacent** if $(u, v) \in E$. We restrict our attention to undirected graphs, i.e. each pair in E is unordered.

Definition 1.2 (Graph isomorphism)**.** Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. A bijective map $\phi : V \to V'$ is called an **isomorphism** from G to G' if both ϕ and its inverse ϕ^{-1} preserves the adjacency of vertices. We say that G and G' are **isomorphic** and write $G \cong G'$.

Definition 1.3 (Embedding)**.** A function $f : X \to Y$ is an **embedding** if and only if $f : X \to f(X)$ is a homeomorphism from X to $f(X)$, where $f(X)$ has the subspace topology from Y.

Definition 1.4 (Isotopy)**.** Suppose that X and Y are topological spaces and $f, g : X \to Y$ are embeddings. An **isotopy** from f to g is a function $H : X \times [0, 1] \to Y$ such that $H(\cdot, 0) = f, H(\cdot, 1) = g$, and $H(\cdot, t)$ is an embedding for each t. If such a function exists, we say that f and g are **isotopic**.

Notice that isomorphic graphs do not necessarily have isotopic embeddings.

Example 1.5. In the case of K_6, as we will show later, there is no way to deform one of embedding of K_6 through space to look like another, without allowing edges to pass through themselves or each other.

Definition 1.6 (Spatial graph and Spatial embedding)**.** Let G be a graph and let f be an embedding of G in S^3. Then we say the image $f(G)$ is a **spatial graph** and f is a **spatial embedding**.

Definition 1.7 (Intrinsically linked and intrinsically knotted)**.** If the image of every embedding of G in S^3 contains a non-trivial link then we say G is **intrinsically linked**, and if the image of every embedding of G in S^3 contains a non-trivial knot then we say G is **intrinsically knotted**.

2. Links and K_6

Theorem 2.1. *K_6 is intrinsically linked.*

In [Sac83] and [CMG83], Sachs, Conway and Gordon proved this theorem independently. Here, we present the combinatorial argument in [CMG83].

Let ϕ be an embedding of K_6 into S^3. Let (C_1, C_2) be an unordered pair of disjoint cycles of K_6. Notice that both C_1 and C_2 contains exactly three disjoint edges.

Claim 2.2. There are 10 distinct pairs of (C_1, C_2).

Proof. The number of possible ways to choose an ordered pair (C_1, C_2) is given by

$$\binom{6}{3} = \frac{6!}{3!3!} = 20 \tag{2.0.1}$$

Hence, the number of ways to choose an unordered pair (C_1, C_2) is simply $20/2 = 10$. □

Let $lk(\phi(C_1), \phi(C_2))$ be the linking number of the embedding of this pair of cycles. Let $\lambda \in \mathbb{Z}_2$ be the sum of $lk(\phi(C_1), \phi(C_2))$ over ten possible pairs mod 2. That is,

$$\lambda = \sum_{(C_1,C_2)} lk(\phi(C_1), \phi(C_2)) \mod 2 \tag{2.0.2}$$

The main idea of the proof is that if $\lambda = 1$ for one of the embedding of K_6, then this embedding contains a nontrivial link. Hence, if $\lambda = 1$ despite how you changes crossings to go from one embedding to another, then K_6 is intrinsically knotted.

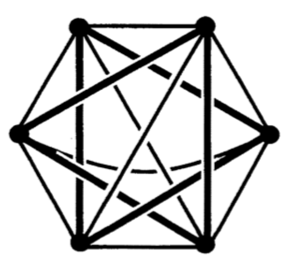
2.1 Crossing change

Claim 2.3. λ is invariant under crossing changes.

Proof. We consider two cases. Let the disjoint pair (C_1, C_2) be given,
Suppose we change a crossing between an edge with itself or two adjacent edges. Then because C_1 and C_2 are disjoint, the crossing must lie entirely within C_1 or C_2. $lk(\phi(C_1), \phi(C_2))$ remains unchanged and thus λ is unchanged.
Suppose we change a crossing between two non-adjacent edges. Let e_1, e_2 be such two edges and without loss of generality, suppose $e_1 \in C_1$ and $e_2 \in C_2$. If we change such a crossing, then we changed the linking number $lk(\phi(C_1), \phi(C_2))$ by ± 1. Notice that there is exactly another pair (C_1', C_2') of disjoint cycles such that $e_1 \in C_1'$ and $e_2 \in C_2'$. Hence, we have that $\sum_{(C_1,C_2)} lk(\phi(C_1), \phi(C_2))$ will change by ± 2 or 0. That is, λ remains invariant. □

2.2 Linked Embedding of K_6

As λ remains invariant under crossing change, if we have $\lambda = 1$ for one of the embedding of K_6, then for all possible embeddings of K_6, it must contains a nontrivial link. As shown in the figure below, this embedding contains 1 link as marked by the bold lines.



2.3 Peterson family ❖

Definition 2.4 ($Y\Delta$-exchange)**.** Let $G = (V, E)$ be a graph and $v \in V$ a vertex of degree 3. Let H be obtained from G by first deleting v and its adjacent edges, and then adding an edge between every pair of neighbours of v. We say that H is obtained from G by $Y\Delta$**-exchange** and that G is obtained from H by ΔY**-exchange**.

Definition 2.5 (Peterson family)**.** The set of all graphs obtained from K_6 by a series $Y\Delta$**-exchange**s and ΔY**-exchange**s is called the **Peterson family**.

There are 7 graphs in the Peterson family. This picture is taken from [O'D10].

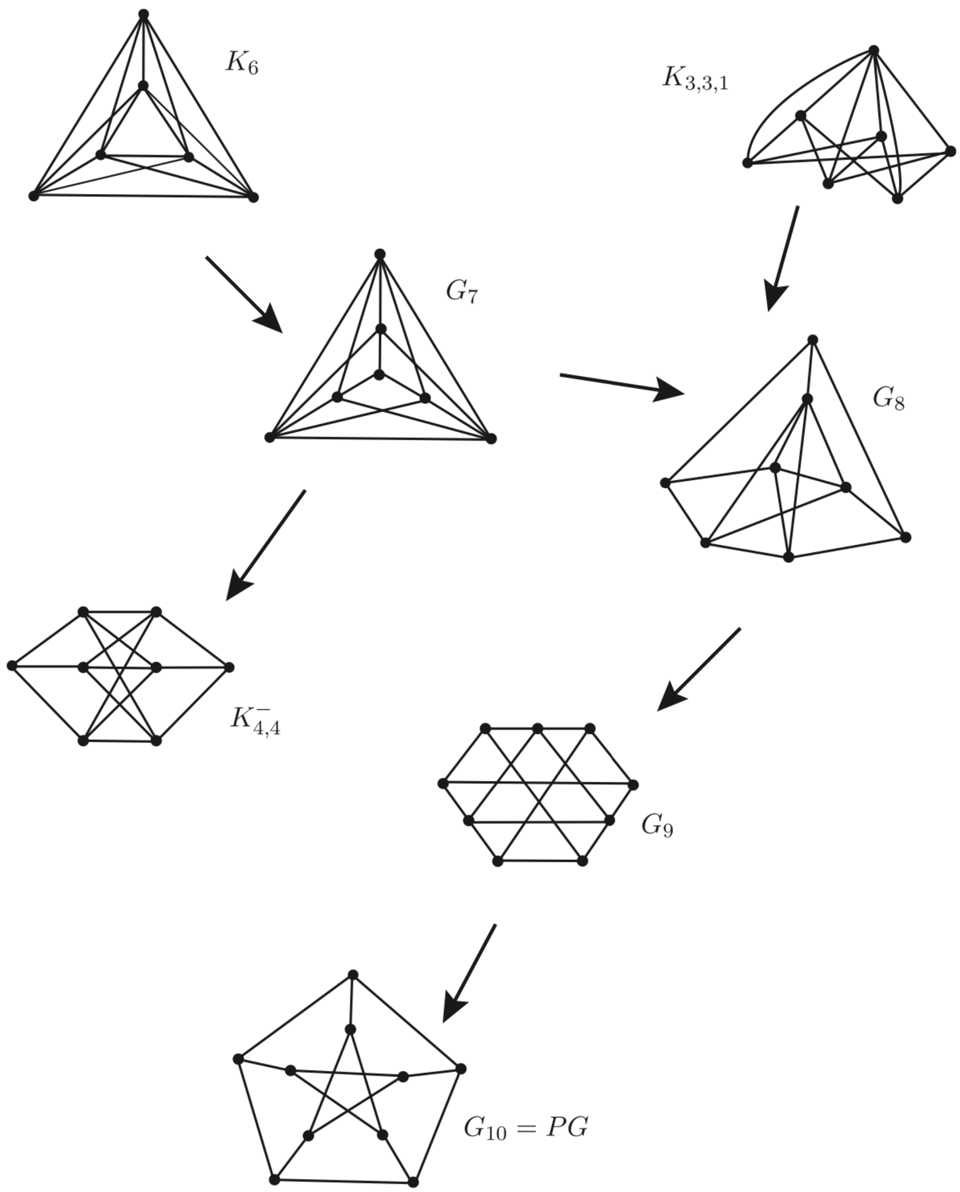
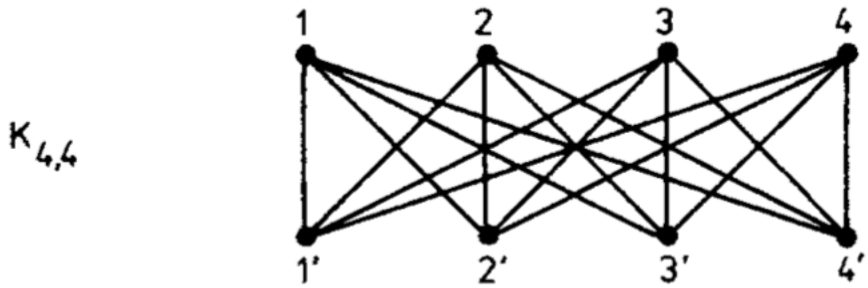


Fig. 1. The graphs of the Petersen family. The arrows indicate a ∇Y-move.

Sachs proved the following using a similar argument we have just shown.

Theorem 2.6 ([Sac83]). *Any graph in the Peterson family is intrinsically linked.*

Furthermore, Sachs has shown that $\lambda(K_{4,4})$ is always 0 and thus if we let $K_{4,4}^-$ be a graph obtained from $K_{4,4}$ by removing edge $(1, 1')$, then $\lambda(K_{4,4}^-)$ is always 1.



Theorem 2.7 ([Sac83]). *Any graph obtained from $K_{4,4}^-$ by a series $Y\Delta$-**exchange**s and ΔY-**exchange**s is intrinsically linked.*

Notice that this is not a surprise as $K_{4,4}^-$ is intuitively "two copies of $K_{3,3,1}$ glued together on the $K_{3,3}$ subgraph". As we will see below in theorem 2.10, this can be formally generalised.

Definition 2.8 (Flat embedding). An embedding ϕ of a graph G in S^3 is **flat** if for every cycle C of G there exists an open disk in S^3 disjoint from $\phi(G)$ whose boundary is $\phi(C)$.

Definition 2.9 (Graph minor). A graph H is a **minor** of another graph G if H can be obtained from G by contracting edges.

Robertson, Seymour and Thomas proved that a graph G has a linkless spatial embedding if and only if it does not a minor in Peterson family.

Theorem 2.10 ([RST93]). *For a graph G, the following are equivalent.*

1. *G has a flat embedding.*
2. *G has a linkless embedding.*
3. *G has no minor in the Peterson family.*

It is intuitive that flat embedding implies linkless embedding. Step 2 to Step 3 is partially proved in theorem 2.6. However, the step from 3 to 2 is very nontrivial to prove.

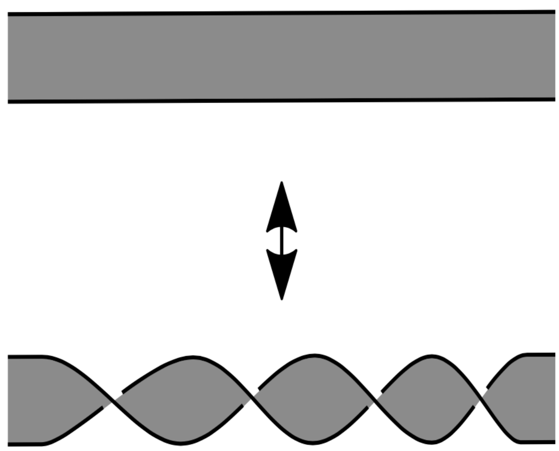
3. Knots and K_7

Theorem 3.1. *K_7 is intrinsically knotted.*

The spirit of the proof is very similar to that of linked-ness of K_6. Before we jump into the proof, we present a different type of invariant.

3.1 Arf Invariant

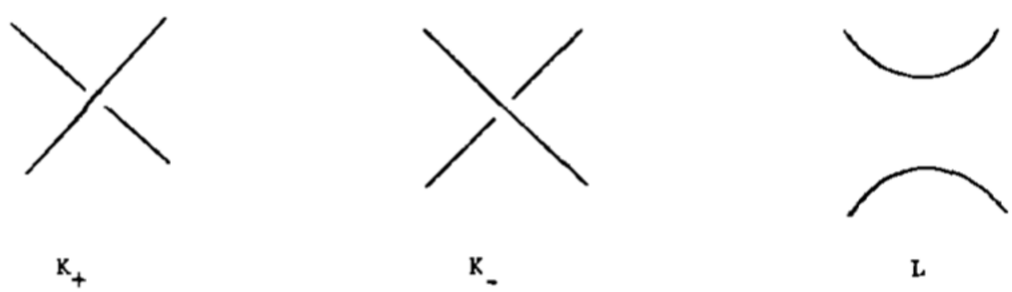
We take the definition of Arf Invariant introduced in [Kau83], [Ada04] and [Man18]. The Arf invariant comes from Seifert surfaces. Namely, let K be a knot and l be a band that is a part of the Seifert surface of the knot K, as shown in the upper part of the figure below. Let K' be the knot obtained from K twisting this band by two full turns, i.e. double twisting, as shown in the lower part of the figure below.



Definition 3.2 (As stated in [Man18])**.** We say that K and K' are **Arf equivalent** and the **Arf invariant** is the complete invariant of the Arf equivalent classes.

Theorem 3.3. *Each knot is either Arf equivalent to the unknot or to a trefoil.*

This theorem allows us to denote $\alpha(K) \in \mathbb{Z}_2$ as $\alpha(K) = 0$ if K is Arf equivalent to the unknot and $\alpha(K) = 1$ otherwise. Let knots K_+, K_- and a two component link $L = L_1 \cup L_2$ be identical except for a small neighbourhood as shown in the figure below.



We state the following lemma without a formal proof as it involves symplectic basis, which is beyond the scope of this note.

Lemma 3.4 ([CMG83])**.** $\alpha(K_+) - \alpha(K_-) = lk(L_1, L_2)$

3.2 Main Proof ❖

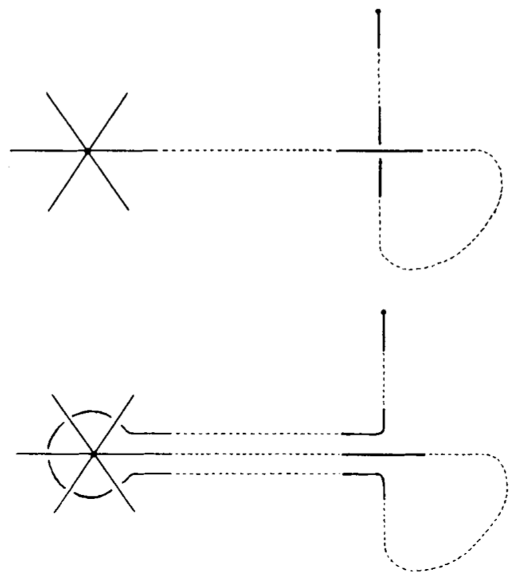
The main proof of theorem 3.1 bears a combinatorial nature. The goal is to show that

$$\sum_{C \subseteq K_7, C \text{ is a Hamiltonian cycle}} \alpha(C) \tag{3.2.1}$$

is invariant under crossing changes. Notice that there are $(7-1)!/2 = 360$ Hamiltonian cycles in K_7.

Proof of theorem 3.1. We consider three types of crossings.

1. A crossing between an edge with itself. Notice that we don't need to consider this case. As shown in the figure below, such a crossing can be replaced by 5 changes of crossings with other distinct edges.



2. A crossing between two adjacent edges A, B, as shown below. Let C be a Hamiltonian cycle that contains A and B and $L = L_1 \cup L_2$ be the link determined by C.

By lemma 3.4, *the change induced by this crossing change is exactly* $lk(L_1, L_2)$. Let F be an edge in C and $F \neq A, F \neq B$. Let $\omega(L_1, F)$ be the number of times that L_1 crosses over F in the projection. Notice that $lk(L_1, L_2) = \sum_F \omega(L_1, F)$ summing over all possible choices of F.
Note that E, A and B cannot have a common vertex since otherwise C is a cycle. If E is adjacent to A or B but not both, there are 6 such cycles C. If E is not adjacent to either A or B, then there are in total $2 \times 3!$ such cycles C. Hence, we have that

$$\sum_C lk(L_1, L_2) \mod 2 = \sum_C \sum_F \omega(L_1, F) \mod 2 = 0 \tag{3.2.2}$$

because for each possible F, there is exactly an even number of repetitions of $\omega(L_1, F)$ that appears in $\sum_C \sum_F \omega(L_1, F)$.

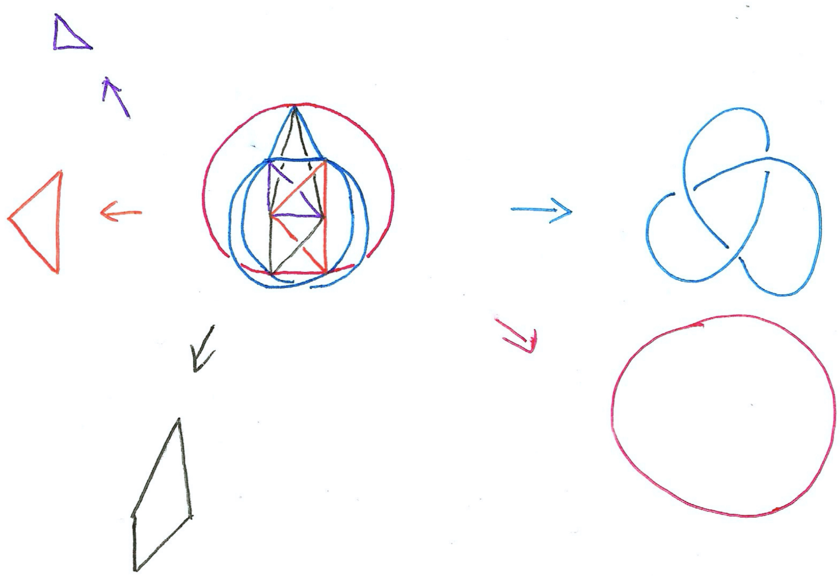
3. We now consider the case that A, B are not adjacent, as shown below. C, L_1, L_2 are defined similarly.

It suffices to show that

$$\sum_C lk(L_1, L_2) \mod 2 = \sum_C \sum_{F_1, F_2} \omega(F_1, F_2) \mod 2 = 0 \tag{3.2.3}$$

for all pairs of edges $(F_1, F_2), F_1 \in L_1, F_2 \in L_2, F_1 \notin \{A, B\}, F_2 \notin \{A, B\}$. Again, the number of Hamiltonian cycles containing A, B, F_1, F_2 and satisfying $F_1 \in L_1, F_2 \in L_2$ is even.

It remains to show that one embedding of K_7 is knotted, as we show below.



$\square$

4. Acknowledgement

Many thanks to UT Math for hosting the Directed Reading Program in Spring 2020. I genuinely appreciate Alexandra Embry for her mentorship for this directed reading project.

4. References

❖

[Ada04] Colin Conrad Adams. *The knot book: an elementary introduction to the mathematical theory of knots.* American Mathematical Soc., 2004.

[CMG83] J. H. Conway and C. McA. Gordon. Knots and links in spatial graphs. *Journal of Graph Theory*, 7(4):445–453, 1983.

[Kau83] Louis H Kauffman. Combinatorics and knot theory. *Contemporary Math*, 20:181–200, 1983.

[Man18] Vassily Olegovich Manturov. *Knot theory.* CRC press, 2018.

[O'D10] Danielle O'Donnol. Knotting and linking in the petersen family. *arXiv preprint arXiv:1008.0377*, 2010.

[RST93] Neil Robertson, P. D. Seymour, and Robin Thomas. Linkless embeddings of graphs in 3-space. *Bulletin of the American Mathematical Society*, 28(1):84–90, Jan 1993.

[Sac83] Horst Sachs. On a spatial analogue of kuratowski's theorem on planar graphs — an open problem. In M. Borowiecki, John W. Kennedy, and Maciej M. Sysło, editors, *Graph Theory*, pages 230–241, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.